SELECTION OF SPATIAL AND TEMPORAL DISCRETIZATION IN WETLAND MODELING

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ABSTRACT

Spatial and temporal discretizations are key factors deciding the optimal use of computer resources in a wetland modeling application. The discretization should be sufficiently fine to describe the solution with a reasonable resolution, and prevent excessive numerical errors. It should not be too fine to prevent the run times becoming excessive. The paper describes a study aimed at understanding the variation of numerical errors and the run times of 1-D and 2-D overland and groundwater flow models, in terms of non-dimensional space and time discretizations.

Fourier analysis of the linearized governing equation is used in the study to obtain analytical expressions for numerical errors and run times. Numerical experiments are carried out with explicit model to obtain results for comparison. The paper describes the use of the results in selecting space and time discretizations for wetland modeling applications.

INTRODUCTION

Overland and groundwater components of flow in wetland models are governed by nonlinear and linear parabolic partial differential equations. The Natural System Model (NSM) and the South Florida Water management Model (SFWMM), (Fennema, et al., 1994, Lal, 1998) are two models based on diffusion flow that are used to simulate flow in the Everglades. Other 2-D overland models using the diffusion flow assumption include the WETFLOW model by Feng and Molz (1997), and the models by Hromadka and Lai (1985). Two dimensional application of the MODFLOW model (McDonald and Harbough, 1984) is based on a similar governing equation. Numerical methods available to solve parabolic equations using rectangular spatial grids include the explicit method, the alternating direction explicit methods (ADE) in the case of the NSM/SFWMM models, the implicit methods as in WETFLOW and MODFLOW models, and the ADI method.

In addition to instability, both explicit and implicit models have to consider restrictions due to excessive numerical errors as well as run times, if proper discretizations are not used. These restrictions are evaluated in this study using analytical methods employing Fourier analysis, and the results are verified using numerical models.

NUMERICAL ALGORITHMS

Two dimensional groundwater flow or overland flow with negligible inertia effects can be expressed as (Hromadka and Lai, 1985, Lal, 1998).

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial H}{\partial y}\right) + S \tag{1}$$

in which, $K = \frac{h^{\frac{5}{3}}}{n_b\sqrt{S_n}}$ for overland flow when the Manning's equation is used; $n_b =$ Manning's coefficient; $h =$ water depth; $S_n =$ water surface slope; S represents source and sink terms. $K =$ hydraulic transmissivity/storage coefficient in the case of groundwater flow. The continuity equation for an arbitrary cell in the finite volume formulation is (Lal, 1998)

$$H_{i,j}^{n+1} = H_{i,j}^{n} + \alpha\, Q_{net}(H^{n+1})\frac{\Delta t}{\Delta A} + (1-\alpha)\; Q_{net}(H^{n})\frac{\Delta t}{\Delta A} \tag{2}$$

in which $\Delta A =$ area of the cell; $Q_{net} =$ net inflow to the cell; $\alpha =$ weighting factor for semi-implicit schemes; $n =$ time step. For a rectangular cell, Q_{net} is given by

$$\begin{aligned} Q_{net} = {} & K_{i+\frac{1}{2},j}(H_{i+1,j} - H_{i,j}) + K_{i-\frac{1}{2},j}(H_{i-1,j} - H_{i,j}) \\ & + K_{i,j+\frac{1}{2}}(H_{i,j+1} - H_{i,j}) + K_{i,j-\frac{1}{2}}(H_{i,j-1} - H_{i,j}) \end{aligned} \tag{3}$$

Explicit and the implicit methods are obtained by using $\alpha = 0$ and 1.0.

ERROR ANALYSIS

Numerical stability and error analysis can be studied using the Von Neuman method. In the analysis, the behavior of the numerical scheme in response to an arbitrary i th harmonic is compared with the behavior of the governing equation with respect to the same harmonic. A harmonic with a dimensionless wave number $\phi = k\Delta x = \frac{i\pi}{N}$ is used in which $k = i\frac{\pi}{L} =$ wave number, and $L =$ length of the solution domain. The time variations of the solution at frequency f is represented using a similar dimensionless parameter $\psi = f\Delta t$ in the discretization. Errors due to temporal and spatial discretizations of the Fourier components of continuous functions which are expressed in terms of wave numbers k and frequencies f can be computed geometrically. The required dimensionless discretization for a given maximum percentage error ε_d is

$$\phi(= k\Delta x) \quad \text{or} \quad \psi(= f\Delta t) = \pi\left(\frac{\varepsilon_d}{143}\right)^{0.3522} \tag{4}$$

For a 1% error in discretization for example, ϕ or ψ of 0.547 is adequate, which means, approximately $\frac{\pi}{\phi} = 6$ discretizations are needed to describe half of a sine wave. Similarly, 3 discretizations ($\phi = 1.05$) bring the error upto 6.3%.

Numerical errors are computed by considering that the 1-D and 2-D analytical solutions are of the form $H = H_0 e^{i(kx-ft)}$, and $H = H_0 e^{i(kx+ky-ft)}$ respectively. k is assumed to be same in both x and y directions in 2-D. The 1-D and 2-D solutions require $f = Kk^2$ and $f = 2Kk^2$, which give a relationships between the frequency of water level variations and the wave numbers of water surface profiles. For a numerical methods expressed using (2) and (3), an analytical expression for the numerical error per timestep can be expressed as a fraction of the amplitude (Hirsch, 1989, Lal, 1998)

$$\varepsilon = 1 - \frac{1-4d(1-\alpha)\beta\sin^2(\phi/2)}{1+4d\alpha\beta\sin^2(\phi/2)}\frac{1}{e^{-d\beta\phi^2}} \tag{5}$$

in which $\beta = \frac{K\Delta t}{\Delta x^2}$ = non-dimensional form of Δt; $d = 1, 2$ for one and two dimensional problems with square grids. Equation (5) can be expanded to give

$$\varepsilon = \pm\frac{d^2\beta^2\phi^4}{2} + \frac{d\beta\phi^4}{12} + \ldots = \pm\frac{d^2k^4K^2\Delta t^2}{2} + \frac{dKk^4\Delta t\Delta x^2}{12} + \ldots \tag{6}$$

in which $+$ and $-$ signs correspond to implicit and explicit models respectively. The cumulative numerical error ε_T after a time T depends on the number of time steps n_t, and the error at each time step ε. ε_T is bound by $n_t\varepsilon$, in which $n_t = T/\Delta t$.

$$\varepsilon_T \approx \frac{\varepsilon}{\beta\phi^2}TKk^2 = \frac{\varepsilon}{d\beta\phi^2}fT \tag{7}$$

where, $k =$ wave number of the harmonic. In the application of this equation, T is the maximum duration over which a given harmonic stays in the computational domain. In the case of a boundary disturbance affecting the interior of the solution, $T = kx/f$ is the time taken for the disturbance to travel a distance x from the boundary. For disturbances created by effects such as the rain, T is roughly the duration of the flood decay. ε_T for explicit, implicit and semi-explicit 1-D and 2-D models are computed using a limited number of terms in the Taylor series of (7). The expressions below and are valid for small ε and ϕ, and use $f = dKk^2$ to relate f to k.

$$\varepsilon_T \quad \text{(expl/impl 1-D)} \quad \approx \quad \frac{fT\phi^2}{2}(\mp\beta - \frac{1}{6}) \tag{8}$$

$$\varepsilon_T \quad \text{(semi-impl 1-D)} \quad \approx \quad fT\left[\frac{\phi^2}{12} - \frac{\phi^4}{12}(\beta^2 - \frac{1}{30})\right] \tag{9}$$

$$\varepsilon_T \quad \text{(expl/impl 2-D)} \quad \approx \quad fT\phi^2(\pm\beta - \frac{1}{12}) \tag{10}$$

$$\varepsilon_T \quad \text{(semi-impl 2-D)} \quad \approx \quad fT\left[-\frac{\phi^2}{6} + \frac{2\phi^4}{3}(\beta^2 + \frac{1}{120})\right] \tag{11}$$

Explicit 1-D and 2-D models require $\beta < 0.5$ and $\beta < 0.25$ respectively, while implicit methods use much larger β (Lal, 1998).

RUN TIME

Computer run time of a model is proportional to the number of cells and the number of time steps. For a 2-D problem, computer run time t_r (cpu) assuming a fixed numerical error target is (Lal, 1998)

$$t_r = \frac{c_r T_p K A k^4}{\beta \phi^4} = \frac{c_r T_p A f^2}{4K\beta\phi^4} \tag{12}$$

in which $A =$ area simulated by the model; $T_p =$ period of simulation; $c_r =$ a computer dependent constant representing run time per cell per time step. $c_r = 3.4 \times 10^{-5}$ and 6.1×10^{-5} for explicit and SOR methods. Equation (12) shows that the run time increases rapidly as ϕ decreases.

NUMERICAL EXPERIMENTS

Two numerical experiments are used to verify the expressions for numerical errors and run times. In the first experiment, a sinusoidal water level fluctuation $H = H_0 \sin(ft)$ was introduced at the boundary of a 1-D explicit model, and the solution was studied. The analytical solution $H = H_0 e^{-kx} e^{i(ft-kx)}$ with $f = 2Kk^2$ is compared with the numerical solution to compute the error. $T = kx/f$ is used in (7) to compute ε_T and provide the following analytical expression.

$$\varepsilon_T(x) = \frac{k\varepsilon}{\beta\phi^2} x \tag{13}$$

Numerical values of $\varepsilon_T(x)$ were computed using simple models.

The second test involves flow over a 161 km $\times$ 161 km square are with an initial water level at the center of $H = [0.4575 + 0.1525\cos(\pi r/r_m)]m$ for $r \leq r_m$ and $H = 0.305m$ otherwise, in which $r =$ radius from the center, and $r_m = 32188$ m (Lal, 1998). The period of simulation is 12 days. Water level at the center is computed analytically and numerically in the test.

RESULTS AND DISCUSSION

Figure 1 shows a comparison of the numerical error per time step computed analytically and numerically for the 1-D explicit method. The results are independent of the actual physical dimensions and parameters. They are presented for three different values of ϕ. The figure shows that the analytical and numerical values agree very closely.

Figure 2 shows the variation of maximum numerical errors and run times of the 2-D overland flow experiment for $\phi = 0.31$. The figure shows that the expressions for run time and error developed in the study agree in trend with values observed in actual test model runs. Details can be found in the paper by Lal (1998).

The steps needed to obtaining Δx and Δt for a hypothetical new 2-D groundwater model are explained using the following example. Assume that the model has to be developed to represent water surface details as fine as 200 m long sine waves (wavelength $\lambda = 200$m. The numerical error at 180 m from the boundary is not to exceed 10% of the amplitude. Assume that the stresses are 1-D, and are introduced at one boundary. The discretization error is not to exceed 6% anywhere. Assume $K = 100m^2/s$ for groudnwater flow.

A. For a $< 6\%$ spatial discretization error, pick $\phi \sim 1.03$ using (4).

B. Compute $k = 2\pi/\lambda = 2\pi/200 = 0.0314$ which gives $\Delta x = \phi/k = 32.8m$. Round off Δx to 30 m, making $k = 0.034$. Using $f = Kk^2$, compute $f = 0.116s^{-1}$. This represents variations of a period 54 s.

C. Compute the travel time of a boundary disturbance using $T = kx/f$ = 53 s. For a 10% error, use (8) to compute β as 0.197 which in turn gives Δt as 1.77 s whe using $\beta = K\Delta t/\Delta x^2$.

D. Check if this Δt is small enough to represent 54 s period waves, using (4). The answer is yes.

E. Equation (12) shows that the run time is 28 s for a running 54 s of the simulation. If the run time is slightly excessive, relax the 10% error in C, and redo the steps. If it is extremely excessive, give up trying to represent 200 m long wave profiles, and redo steps from B with the new k.

CONCLUSIONS

Analytical expressions are derived for the maximum numerical errors and run times of various 1-D and 2-D numerical methods using non-dimensional space and time discretizations ϕ and β. Numerical experiments show the validity of the expressions for both overland and groundwater flow. An example is shown to illustrate the steps needed to determine the discretizations for a new groundwater flow model.

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Figure 1: Variation of error with spatial resolution for sine waves.

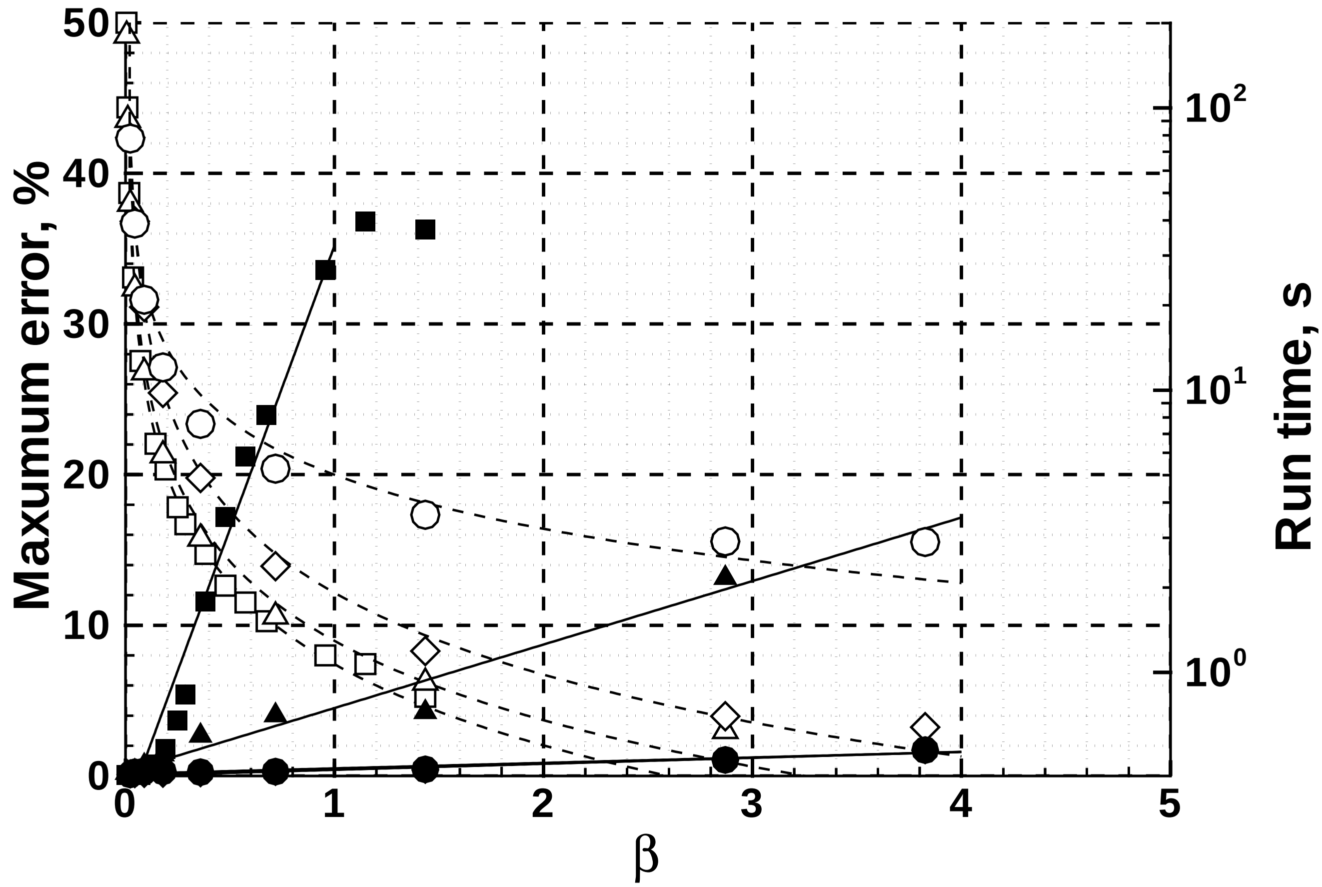

Figure 2: Variation of maximum errors (solid symbols) and run times (empty symbols) with β using all four models when $\phi = 0.31$. Solid and dashed lines show the trends in errors and run times respectively. Note that ADI and SOR symbols coincide. Square symbols = explicit method; triangular = ADE; diamond = ADI; circle = SOR.