CALIBRATION OF RIVER BED ROUGHNESS

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Singular value decomposition (SVD) is used to calibrate the Manning's roughness coefficients in a 1-D unsteady flow model of the Upper Niagara River. The method is used to solve for the parameters after formulating the calibration problem as a generalized linear inverse problem. SVD is useful in solving under-determined, over-determined or even-determined problems, and can provide information to compute matrices describing parameter resolution, covariance and correlation. This information is useful in identifying the important parameter groups in the model. Calibration is repeated with different numbers of parameter groups to determine the variation of the output error and uncertainty of the parameters with the parameter dimension. For purposes of comparison, the model with a selected group of parameters is calibrated using Gauss-Newton method and minimax methods. The study shows the relationship of the parameters to the geometric layout of the river and the gaging stations.

INTRODUCTION

Problems involving mathematical models are categorized as direct problems and inverse problems. In direct problems, everything about the model is known, and the objective is to find the system output for a given input. Inverse problems are classified as identification, detection and state reconstruction problems. Calibration is an inverse problem associated with identification, and is used to determine unknown constants or parameters in a model.

Direct or explicit parameter determination is not possible in many nonlinear problems, including unsteady river flow networks. Indirect methods based on an output error criterion are commonly used to determine bed roughness parameters iteratively. Algorithms used in calibration generally include methods based on mathematical programming, optimal control or trial and error techniques. A summary of calibration methods used in groundwater problems is given in the text by Willis and Yeh (1987). Although there is theoretically an infinite number of spatially dependent

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bed roughness parameters in a river network, only a finite number exists in a computer model due to spatial discretization used with finite difference or finite element methods. Out of this finite number, only a limited number can be identified in most inverse problems (Beck, 1987). The current study investigates the use of Singular Value Decomposition (SVD) to study parameters in a river network model. The results are also used to form parameter groups to simplify the problem and avoid getting a severely underdetermined systems. The study shows that some of the parameter groups can also be associated with the geometric layout of the river network and the placement of observation stations.

When identifiability is not a problem, popular methods of calibration used in the past for open channel problems include methods by Becker and Yeh (1972), Yeh and Becker (1973), and Fread, et al. (1978). Related applications include calibration of surface irrigation parameters by Katopodes et al. (1990). Error criteria commonly used in the calibration are based on minimization of the sum of square of the errors in calculations, the maximum absolute error or the bias. Fread (1978) used a method based on minimization of the bias, which required breaking down of the river into a number of single channel reaches before calibrating each reach separately. The Kalman filtering method used by Chiu, et al. (1978) is also a valuable tool in calibration, adopted from optimal control theory.

The SVD method has been previously used by Wiggins (1972) and Uhrhammer (1980) to calibrate seismologic parameters. Text by Meneke (1984) gives a description of inverse methods used in solving geophysical problems. The objectives of this study includes determination of the structure, accuracy, and the optimum parameter dimension in the river network model. Parameter covariance and correlation matrices are also determined in the study. The calibration is repeated using optimization methods based on the Gauss-Newton method and the minimax methods for comparison of results .

THEORETICAL CONSIDERATIONS

The theoretical derivation of the equations required for calibration assumes that the numerical model to be calibrated can be represented as a continuous function of the model parameters locally within the useful range of the variables. If y_j^k are the values of state variables or water levels simulated by the model at times k at locations j, and if Y_j^k are the physical observations of the same variables, the objective of the calibration is to find parameters x_i, $i = 1, 2, \ldots, n$ that minimize the errors $\varepsilon_j^k = y_j^k - Y_j^k$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, l$, using a specified criterion. $n =$ number of parameters; $m =$ number of observation stations; and $l =$ number of time steps.

The influence coefficient method (eg., Becker and Yeh, 1972) uses parameter perturbations to define influence or sensitivity coefficients.

$$a_{ji}^k = \frac{\partial y_j^k}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{y_j^k(x_i + \Delta x_i) - y_j^k}{\Delta x_i} \quad \text{for each} \quad i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m,\ k = 1, 2, \cdots, l \quad (1)$$

in which, $a_{ji}^k =$ sensitivity of the j th water level with respect to i th parameter at time k; $x_i = i$ th parameter; $k =$ time step number; $l =$ number of time steps. Using matrix notation, (1) can also be expressed as

$$\blacksquare \mathbf{y}_{m\times 1}^k = \mathbf{A}_{m\times n}^k \blacksquare \mathbf{x}_{n\times 1}^k \quad (2)$$

in which, the matrix $\mathbf{A}_{m\times n}^k$ is called the influence matrix. The superscript k is used to identify the time step.

Numerical computation of influence coefficients is carried out by perturbing each of the parameters by a small amount Δx_i. For the current calibration of Manning's coefficients, Δx is 0.001. If n parameters are calibrated, the model is run $n+1$ times to obtain all the $m \times n$ elements of the sensitivity matrix.

Generalized linear inverse problem

The method used by Wiggins (1972), Ward et al. (1973) and Uhrhammer (1980) to solve linear inverse problems is used in the current study. Since there is a time series of errors at every gage, bias is used as the error indicator, and there are m error indicators for the problem.

$$\varepsilon_{sj} = \frac{1}{l} \sum_{k=1}^{l} (y_j^k - Y_j^k) \quad \text{for} \quad j = 1, 2, \ldots, m \quad (3)$$

in which, ε_{sj} = average error or bias at gage j, which is ε_s in vector form. The corrections in parameters $\Delta\mathbf{x}$ needed to make $\varepsilon_s = 0$ can be determined by equating average $\blacksquare\mathbf{y}$ of (2) to -ε_s and solving for $\blacksquare\mathbf{x}$. The resulting system of linear equations can be expressed as (Wiggins 1972, Ward et al. 1973).

$$\frac{1}{l}\sum_{k=1}^{l} a_{jp}^{k}\Delta x_p = -\frac{1}{l}\sum_{k=1}^{l}(y_j^k - Y_j^k) \quad \text{for} \quad j = 1,2,\ldots m, \quad p = 1,2,\ldots n \tag{4}$$

or, in matrix form,

$$\mathbf{A}_s\Delta\mathbf{x} = -\varepsilon_s \tag{5}$$

in which,

$$\mathbf{A}_s = \frac{1}{l}\sum_{k=1}^{l}\mathbf{A}^k, \qquad \varepsilon_s = \frac{1}{l}\sum_{k=1}^{l}\varepsilon^k \tag{6}$$

The corrected set of parameters to be used in the next iteration are

$$\mathbf{x}_{r+1} = \mathbf{x}_r + \rho_r\Delta\mathbf{x}_r \tag{7}$$

subjected to

$$\mathbf{x}_{r,lb} \leq \mathbf{x}_{r+1} \leq \mathbf{x}_{r,ub} \tag{8}$$

in which r is the iteration number; ρ_r = parameter used to control the step size; ub, lb are specified upper and lower bound values. $\rho_r \approx 0.8$ is used in the current study. The iterative procedure is continued until $||\varepsilon_s||_2$ converges, and $||\blacksquare\mathbf{x}||_2$ becomes small $(||\blacksquare\mathbf{x}/\mathbf{x}|| < \delta)$ in which δ = machine precision of the computer or a larger value depending on the required parameter tolerance.

The m equations shown in (5) can be solved easily using methods such as Gaussian elimination if $m = n$ and $\det\mathbf{A} \neq 0$. When $m > n$, the problem is overdetermined, and the least squares method leads to the solution $\mathbf{x} = -(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\varepsilon_s$ as long as $\det\mathbf{A}^T\mathbf{A} \neq 0$. When $m < n$, the problem is underdetermined, and the generalized or Penrose inverse solution is $\mathbf{x} = -\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\varepsilon_s$ as long as $\det\mathbf{A}\mathbf{A}^T \neq 0$ (Noble and Daniel, 1975). None of these solution techniques can be used with rank deficient or ill-conditioned matrices. The sensitivity matrix becomes rank deficient if there is at least one parameter that has no significant influence on the any of the selected observations, or if there is at least one observation station that is not sufficiently affected by any of the parameters. Singular Value Decomposition (SVD) is capable of solving under, over, even or mixed determined problems as explained in detail by Meneke (1984).

SVD is based on the key theorem that a matrix $\mathbf{A}_s$ can be decomposed into three matrices $\mathbf{V}$, Λ and $\mathbf{U}$ such that

$$\mathbf{A}_s = \mathbf{U}\Lambda\mathbf{V}^T \tag{9}$$

in which $\mathbf{U}$ and $\mathbf{V}$ are $m \times m$ and $n \times n$ matrices of orthogonal singular vectors and $\Lambda = m \times n$ diagonal matrix of singular values of $\mathbf{A}_s$. Texts by Noble and Daniel (1975), Forsythe et al. (1977) and Press, et at., (1989) give detailed information about the method. With symmetric matrices, SVD gives eigenvalues and eigenvectors.

The physical meaning of the terms and expressions of the SVD method are explained using (9) and (10). SVD essentially diagonalizes the sensitivity matrix. After the decomposition, singular vectors of length n formed by the columns of $\mathbf{V}$ give coefficients of the linear combinations of old parameters giving rise to new independent parameter groups. Singular vectors of length m formed by columns of $\mathbf{U}$ give coefficients of the linear combinations of observations forming new observations groups. The new independent parameter groups are related to groups of observations through the diagonal elements in Λ. Both newly formed observation and parameter groups are independent of each other because of the orthogonality of matrices $\mathbf{V}$ and $\mathbf{U}$. Singular values forming the diagonal sensitivity matrix Λ relate these parameter groups to observations groups. The number of nonzero singular values q in Λ is the rank of $\mathbf{A}_s$. It gives the maximum number of independent parameter groups that can be identified in the model.

SVD can be used to solve the system of equations shown in (5). The solution method replaces the previously mentioned methods for $m = n$, $m < n$, $m > n$ or mixed determined problems. It works even if the matrices involved are singular (Meneke, 1984). The solution to the system of equations shown in (5) is obtained by decomposing $\mathbf{A}_s$ into $\mathbf{A}_s = \mathbf{U}\Lambda\mathbf{V}^T$. (5) then becomes

$$\Lambda\,\Delta\mathbf{z} = -\mathbf{d} \tag{10}$$

in which,

$$\Delta\mathbf{z} = \mathbf{V}^T\Delta\mathbf{x}, \qquad \text{and} \qquad \mathbf{d} = \mathbf{U}^T\varepsilon_s \tag{11}$$

The solution to (10) can be written as

$$\Delta\mathbf{x} = -\mathbf{V}[(\frac{1}{\lambda})](\mathbf{U}^T\varepsilon_s) \tag{12}$$

in which λ are the diagonal elements of Λ. If at least one λ_i is zero or close to zero such that the value is dominated by the roundoff error, the matrix is singular. When this happens, $1/\lambda$ values corresponding to λ falling below a small cutoff value are all replaced by zeros (Forsythe, et al. 1975, Press, 1989). It can be shown that SVD determines the $\blacksquare\mathbf{x}$ for a problem that minimizes $||\blacksquare\mathbf{x}||_2$ and $||\varepsilon_s||_2$. The solution under these conditions is unique, and includes the Gaussian, least square and Penrose cases mentioned before.

The cutoff level or the smallest singular value λ_{min} in (12) is useful in controlling the data errors in the solution. If it is small, $1/\lambda$ becomes too large, and errors in the solution will be blown out of proportion. In a non singular matrix, λ_{min} is related to the condition number, $\lambda_{max}/\lambda_{min}$. A matrix is ill-conditioned if it is too large, or the reciprocal, $\lambda_{min}/\lambda_{max}$ approaches floating point precision δ. For single precision operations, $\delta \approx 10^{-6}$ for most computers and the minimum cutoff possible is $\delta \times \lambda_{max}$. Since data error can cause unnecessary error magnifications at small cutoffs, a δ value of 10^{-3} was used in the current problem. Even larger values (10^{-2}) were found to be sufficient during initial iterations. Methods of smoothing the cutoff have been discussed Wiggins (1972) and Uhrhammer (1980).

The **U** and **V** matrices created by SVD can give additional information about the parameter behavior. The following expression gives an idea about the parameter resolution in the computer model (Wiggins, 1972):

$$\text{Parameter space resolution,} \quad \mathbf{R} = \mathbf{V}\mathbf{V}^T \tag{13}$$

Relative size of the elements of **R** show relative resolution or independence of the corrections to the parameters. If the parameters are finely resolved, this matrix is expected to be close to unity. If the parameters are not individually resolvable but are resolvable as groups, the problem is characterized by compact resolution, showing square blocks or groups of dominant elements in the matrix. Covariance of the estimated parameters is defined as (Willis and Yeh, 1987)

$$\Upsilon = \text{cov}(\ \hat{\mathbf{x}}) = E\left\{(\bar{\mathbf{x}} - \hat{\mathbf{x}})(\bar{\mathbf{x}} - \hat{\mathbf{x}})^T\right\} \tag{14}$$

in which Υ is the parameter covariance matrix which can be computed as $\sigma^{\ 2}(\mathbf{A}_s^T\mathbf{A}_s)^{-1}$ when the determinant of $(\mathbf{A}_s^T\mathbf{A}_s)^{-1}$ is not zero. $\sigma^2 =$ error variance at the output. When the matrix is ill-conditioned, this method fails. However, the following expression derived using the results of SVD

gives the covariance matrix for both singular and non-singular sensitivity matrices (Wiggins, 1972, Uhrhammer, 1980):

$$\text{Covariance,} \quad \Upsilon^2 \quad = \mathbf{V}\frac{\sigma^2}{\Lambda^2}\mathbf{V}^T \tag{15}$$

A value of 1 is used for σ^2 to obtain the relative values in the matrix elements. Uhrhammer (1980) referred to the covariance matrix as the uncertainty matrix, and used a damping parameter to damp out the effects of small singular values on $1/\lambda$ during a sharp cutoff. In the current study, a sharp early cutoff λ_{min} $(= 10^{-3})$ as suggested by Wiggins (1972) is used for simplicity.

The covariance matrix can be used to obtain the correlation matrix (Uhrhammer, 1980)

$$\text{Correlation among parameter corrections,} \quad \rho^2_{i,j} = \frac{\Upsilon^2_{i,j}}{\Upsilon_{i,i}\Upsilon_{j,j}} \tag{16}$$

Correlation among parameter corrections is useful in determining the dependence of parameters and parameter groups. If there are parameters that are dependent as a group, and as a group if they are independent from other groups reflecting a block form in the matrix, calibration can be simplified by casting these groups as single parameters. The correlation matrix, resolution matrix, and $\mathbf{V}^T$ are used in the current study to understand and isolate the groups of parameters. These groups can also be related to the layout of the river network and gage positions.

Gauss-Newton method

The Gauss-Newton method as discussed by Willis and Yeh (1987) considers the identification problem as an optimization problem aimed at minimizing output errors. This method cannot be used with underdetermined systems $(m < n)$ or systems with rank $q < n$. It is used in the study as an alternative method to calibrate groups of parameters selected to satisfy these conditions. When using the method, the objective is to determine parameters $x_i, i = 1, 2, \ldots, n$ such that the summation of the squares of the errors is a minimum, subjected to parameter constraints given by (8). The sum of squares of errors is defined as

$$S = \sum_{j=1}^{m}\sum_{k=1}^{l} w_{j,k}(y_j^k - Y_j^k)^2 \tag{17}$$

over all the gages and all the time steps. The $w_{j,k}$ are weighing factors, which are assumed to be 1 for the current problem. The parameter corrections that minimize S can be shown to be (Yoon and

Yeh 1976).

$$\mathbf{C}\Delta\mathbf{x} = -\mathbf{D} \tag{18}$$

in which elements i, j of $\mathbf{C}$ and $\mathbf{D}$ are given by

$$c_{i,j} = \sum_{k=1}^{l}\sum_{p=1}^{n} a_{p,i}^{k} a_{p,j}^{k}; \qquad d_i = \sum_{k=1}^{l}\sum_{p=1}^{n} a_{p,i}^{k}(y_p^k - Y_p^k) \tag{19}$$

The $n \times n$ system of linear equations is solved to obtain the corrections $\Delta x_i,\ i = 1, 2, \cdots, n.$ The parameters are updated using (7), in which ρ_r can be assumed. Optimal values of ρ_r can be determined by the quadratic interpolation method suggested by Yoon and Yeh (1976). Iterations can be terminated when S converges and the parameter corrections are too small to make physical sense, or when $||\blacksquare\mathbf{x}/\mathbf{x}|| \approx$ machine precision.

Minimax method

The objective of the minimax method used in the current study is to determine the parameters $x_i,\ i = 1, 2, \cdots, n$ that minimize the sum of the absolute values of the maximum errors at each of the gages. The maximum errors $e_1, e_2, \cdots e_m$ at gages $1, 2, \cdots m$ at any time during the entire period are

$$e_i = \max(|\varepsilon_i^k|), \quad k = 1, 2, \ldots l, \quad i = 1, 2, \ldots m \tag{20}$$

The objective function selected is

$$\min e_1 + e_2 + \cdots + e_m \tag{21}$$

If the parameters $x_i,\ i = 1, 2, \cdots, n$ are subjected to small changes $\Delta x_i,\ i = 1, 2, \cdots, n,$ then simulated values $y_i^k,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, l,$ change to $(y_i')^k = y_i' + \Delta y_i^k,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, l.$ Expressing Δy_i^k using (1), y_i^k can be expressed as

$$(y_i')^k = y_i^k + \sum_{j=1}^{n} a_{ij}\Delta x_j \quad i = 1, 2, \cdots, m,\ k = 1, 2, \cdots, l \tag{22}$$

The constraints used to determine maximum error can be set up by confining the new errors after a parameter correction to be e_i for all stations. This condition is given by

$$|(y_i')^k - Y_i| \le e_i, \quad \text{for} \quad i = 1, 2, \cdots, m, \quad \text{and} \quad k = 1, 2, \cdots, k \tag{23}$$

Substituting for $(y_i')^k$ using (22), (23) can be written as

$$\sum_{j=1}^{n} a_{ij}^k \Delta x_j - e_i \quad \leq \quad Y_i^k - y_i^k \quad \text{for } i = 1,2,\cdots,m, \text{ and } k = 1,2,\cdots,l \tag{24}$$

$$-\sum_{j=1}^{n} a_{ij}^k \Delta x_j - e_i \quad \leq \quad -Y_i^k + y_i^k \quad \text{for } i = 1,2,\cdots,m, \text{ and } k = 1,2,\cdots,l \tag{25}$$

Other conditions can also be included to keep the solution within limits. Upper and lower bounds can be set to the parameters using

$$x_i + \Delta x_i \leq \quad x_{i,ub} \quad i = 1,2,\ldots n \tag{26}$$

$$x_i + \Delta x_i \geq \quad x_{i,lb} \quad i = 1,2,\ldots n \tag{27}$$

in which, $x_{i,ub}$ and $x_{i,lb}$ are the upper and lower bounds of parameters in (8). The objective function with $2 \times l \times m + 2 \times n$ constraints form a linear programming problem, which is solved for the corrections Δx_i, $i = 1,2,\cdots,n$ using the simplex algorithm. Parameters are updated using (7) until the corrections are too small to make physical sense or $||\blacksquare\mathbf{x}/\mathbf{x}|| \approx$ machine precision.

APPLICATION TO THE UPPER NIAGARA RIVER

The model to be calibrated solves the Saint Venant equations using the four point implicit method. The numerical method is similar to that used by Potok and Quinn (1979). The Saint Venant equations consist of the following continuity and momentum equations:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{28}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}(\frac{Q^2}{A}) + gA\frac{\partial h}{\partial x} + \frac{p_b \tau_b}{\rho} = 0 \tag{29}$$

in which, $Q =$ discharge; $A =$ flow area; $h =$ water surface elevation; $x =$ distance along the river; $t =$ time; $\rho =$ density of water; $p_b =$ wetted perimeter; $\tau_b =$ shear stress at channel bottom $= (\rho g Q^2 n_b^2)/(A^2 R^{(1/3)})$; $R =$ hydraulic radius $= A/p_b$; $p_b =$ wetted perimeter; $n_b =$ Manning's coefficient. n_b values for the discretized river sections are the unknown parameters in the model to be calibrated.

The map of the Upper Niagara River is shown in Fig. 1. In the map, the term NYPA stands for New York Power Authority hydropower intake, and SAB stands for the Sir Adams Beck power

plant intake on the Canadian side. The river network is discretized into 27 sections. The problem involves calibration of the ($n = 27$) unknown Mannings coefficients to minimize the errors at all the gaging stations. Hourly water levels between Feb 13, 1989, and Feb 16, 1989, are used in the following sample calibration to obtain Manning's coefficients for the range of water levels considered. A complete calibration requires data for a longer period of time, covering a wider range of water levels. The list of 3 stations with known discharges and 9 stations with known water levels available to run the model is shown in Fig. 1. Water level gages are marked as G1 ... G9 and known discharges marked as Q10 ... Q12. The upstream end water level at Fort Erie gage G1 is specified as the upstream boundary condition, and the downstream end discharges Q10, Q11 and Q12 at the control dam and intakes are specified as the downstream boundary conditions. Water level gage G2 is not used because its value is very close (± 0.05 ft) to the value of G3, and both are represented by only one node in the 1-D model. The remaining 7 gages marked G3-G9 are used in the calibration. In the study, the observations and simulations at these gages are simply referred to as Y_i and y_i, $i = 1, \ldots, m$ in which $m = 7$.

Calibration of individual model parameters

The generalized linear inverse formulation and singular value decomposition are used to calibrate 27 bed roughness values $n_1, \ldots n_{27}$ of the 27 river sections in the 1-D hydrodynamic model. SVD is needed in this case because there are only 7 observation stations or 7 equations to solve (5) to determine 27 unknown parameters. All the Manning's coefficients are assumed as 0.025 in the first iteration.

Figure 2 shows the sensitivity matrix $\mathbf{A}_s$ indicating large sensitivities of n_2 and n_3 to almost all the water level gaging stations. In the figure, the circle diameters correspond to the sizes of the elements. n_2 and n_3 correspond to the rapids sections of the upper Niagara river where the drop in water level is about 2 m and flow velocities reach approximately 2.6 m/s.

Figure 3 shows the convergence of a few selected stable parameters n_2, n_6, n_{11}, n_{13}, and n_{16}, and the rapid reduction of the error variance in water levels with increasing number of iterations. The rapid convergence indicates that the method is capable of minimizing the bias error as intended. Figure 3 shows that the method also reduces the mean square error during calibration. Since the

problem is underdetermined, the solution shown in the second row of Table 1 is nonunique and has a large random error. Parameter error variance is large for relatively insensitive parameters as shown later in Fig. 7. Even if the insensitive parameters have a large variance, the model gives a good error reduction as shown in Fig. 4 because the model output is not affected by them.

The set of 27 parameters with large errors has a limited use. However, the by-products of the method such as the singular values and singular vectors can be used to understand the parameters of the model in great detail as explained before. Three of the new linearly independent parameter corrections obtained using $\Delta\mathbf{z} = \mathbf{V}^T\Delta\mathbf{x}$ of (11) are shown below.

$$\Delta n_1^* \approx 0.712\Delta n_3 + 0.687\Delta n_2 + 0.093\Delta n_4 + 0.043\Delta n_{11} + \ldots, \quad \lambda_1 = 81210 \tag{30}$$

$$\Delta n_2^* \approx 0.594\Delta n_{11} + 0.364\Delta n_{16} + 0.317\Delta n_{21} + 0.253\Delta n_{10} + \ldots, \quad \lambda_2 = 2269 \tag{31}$$

$$\Delta n_3^* \approx 0.622\Delta n_{16} + 0.406\Delta n_{17} + 0.346\Delta n_{21} + 0.229\Delta n_{24} + \ldots, \quad \lambda_3 = 1572 \tag{32}$$

in which, Δn_1^*, $\Delta n_2^*, \ldots$ are the components of the vector $\Delta\mathbf{z}$ representing new independent parameter variations. These variations are made up of linear combinations of old parameter variations Δn_1, $\Delta n_2 \ldots$. The coefficients in front of $\Delta n_1, \Delta n_2 \ldots$, which form Δn_i^*, are obtained from the columns of $\mathbf{V}$. Variations of the new parameters, Δn_i^* are related to variations in the model output, Δh_i^*, through the relationship $\Delta h_i^* = \lambda_i \Delta n_i^*$, which is similar to (10) when the model output variation is replaced by a negative output error. The singular values λ_i of the model 81210, 2269, 1572, 827, 782, 476 and 22, (nearest integer) show the domination of the calibration by a small number of parameter groups. The singular values show that the rank of the (7×27) sensitivity matrix is 7 and that only 7 independent parameter groups can be identified in the problem.

The singular vectors of $\mathbf{V}$ written as (30)...(32) show the three most important parameter groups. Singular values λ_i provide estimates of the sensitivities. The equations show that adjustment of the group of parameters, consisting of mainly n_3 and n_2 as indicated by the strong coefficients 0.712 and 0.687, can take care of about 93 % of the overall calibration. Parameters n_2 and n_3 represent the narrow rapids section of the river that has a controlling effect on the overall water levels. Parameters n_{11}, $n_{16} \ldots$ form another independent group.

The linear combinations of variations of the model output are related to $\Delta n_1^*, \Delta n_2^* \ldots$ through the equation $\blacksquare\mathbf{h}^* = \mathbf{U}^T\blacksquare\mathbf{h}$.

$$\Delta h_1^* \approx -0.544\Delta h_9 - 0.458\Delta h_7 - 0.457\Delta h_8 - 0.414\Delta h_6 + \ldots \quad \lambda_1 = 81210 \tag{33}$$

$$\Delta h_2^* \approx 0.627\Delta h_4 + 0.607\Delta h_3 + 0.287\Delta h_5 - 0.170\Delta h_8 + \ldots \quad \lambda_2 = 2269 \tag{34}$$

$$\Delta h_2^* \approx -0.930\Delta h_5 + 0.265\Delta h_4 + 0.210\Delta h_3 + 0.110\Delta h_9 + \ldots \quad \lambda_3 = 1572 \tag{35}$$

in which, Δh_1^*, $\Delta h_2^*, \ldots$ are the elements of vector $\blacksquare\mathbf{h}^*$ made of linear combinations of variations of water levels $\Delta h_1, \Delta h_2, \ldots, \Delta h_7$ which correspond to gages G3-G9. (33)-(35) show that the new parameter n_1^* uses simulation errors in water levels h_9, h_7, h_8, $h_6 \ldots$ for calibration, and parameter n_2^* uses errors in h_4, $h_3 \ldots$ for calibration. Approximate equality of coefficients in (33) shows that all the gages have the same information required to calibrate n_1^*. Parameter n_2^* however requires information mostly from h_4 and h_3 (G6 and G5) as shown by coefficients 0.627 and 0.607.

Figure 5 shows the parameter resolution matrix obtained using (13). In the figure, parameter corrections in groups $n_6 \ldots, n_{14}$ and $n_{16}, \ldots, n_{24}$ are the least resolved. n_{27} is the most resolved, because there are two gages at both ends of the reach. Parameters within a long river reach without gaging stations in the middle are not sufficiently resolved, and behave as a group. Any further resolution of parameters within these zones is not possible because observation points cannot distinguish between roughnesses of any two sections within a group if the group lies between two gages. It is also unnecessary to resolve the parameters any further unless the model is used to estimate the water levels at the interior points.

The accuracy of the parameter corrections is of the order $\frac{1}{\lambda_{min}}$ as shown by (12). If a small cutoff value λ_{min} is selected, the parameters will have larger errors. If a large cutoff is selected, only a few parameter groups will be selected as explained before. With a very large cutoff, the number of parameters may not be adequate to represent the physical system, and sufficiently minimize the errors. The next sections includes determination of an optimal parameter dimension based on the parameter uncertainty and the least square output error.

The correlation matrix in Fig. 6 shows the group behavior of the parameters. The figure shows the strong correlation among parameter groups $n_6 \ldots n_8$, $n_9 \ldots n_{12}$, $n_{13} \ldots n_{15}$, $n_{16} \ldots n_{18}$, $n_{19} \ldots n_{24}$, $n_{25} \ldots n_{26}$. The map in Fig. 1 shows that these groups correspond to river reaches with water level gages at both ends. Next sections describe the calibration using parameter groups.

Calibration of parameter groups

Calibration of the individual parameters in the previous section shows that they tend to group together as indicated by the correlation, resolution and **V** matrices. In the SVD method explained

earlier, singular vectors in $\mathbf{V}$ could implicitly make use of these groups. However, 27+1 computer runs were needed for one iteration of the calibration. In this section, in order to reduce the number of runs and obtain more stable parameters with low error variances, 7 new groups of parameters are created from the original parameters using the information available from the previous section. A single parameter value is assigned to all the parameters in a group. The parameter groups created in this manner are shown in Table 2 and are marked as $m_1 \ldots m_7$. The number of groups is limited to 7 because it is the rank of $\mathbf{A}_s$.

To study the effect of the number of parameter groups, or parameter dimension on the output error and parameter uncertainty, the 7 parameter groups are combined again with each other to form smaller numbers of parameter groups. Column 4 of Table 2 for example shows how parameters $n_1 \ldots n_{27}$ are combined to form 4 groups. Figure 7 shows the variation of the output error and parameter uncertainty with different numbers of parameter groups. The mean square bias error is used to measure the output error in which bias is the average difference between the observed and simulated water levels. An approximate estimate of the parameter error or uncertainty is computed using the expression σ^2/λ_{min}^2, which is approximately equal to the infinite norm of the covariance matrix. Figure 7 shows that the parameter uncertainty increases and the output error decreases with increasing number of parameter groups. The figure also shows that the output error variance does not reduce much when there are more than 3 parameters. With more than 3 parameters, the parameter uncertainty increases. The figure shows the existence of an optimal parameter dimension of about 3 for the problem where both the output error and parameter uncertainty are low.

SVD with 7 parameter groups shown in Table 2 gave a mean square bias error of $7.23 \times 10^{-4}\, m^2$ after 7 iterations showing that the calibration is successful even if these handpicked parameter groups are not exactly the singular vectors in (30)-(32). The condition number of the 7×7 group sensitivity matrix is 150198/23.8, which shows that the matrix is nonsingular, and it is possible to use the Gauss-Newton method for comparison. The parameter resolution and correlation matrices of the 7 parameter group calibration (not shown) indicate that the handpicked parameter groups are sufficiently resolved and independent.

The third and fourth rows of Table 1 show the values of parameters obtained by calibrating the handpicked 7 and 3 groups of parameters using SVD. All these are even determined problems that can also be solved using Gaussian elimination. The value of a single parameter that fits the

entire river network is found to be 0.03022, with an output mean square bias error of $6.1 \times 10^{-2}\ m^2$.

Calibration using Gauss-Newton and Minimax methods

The Gauss-Newton and minimax methods are also used to calibrate the same 7 handpicked parameter groups for comparison. The final Mannings coefficients after 5 iterations of calibration using both methods are shown in Table 3 along with the sum of error variances and mean square bias errors. The largest correction required on the Mannings coefficients after 5 iterations of the Gauss-Newton method is approximately 0.00002, and the largest correction required after 8 iterations of the minimax method is approximately 0.00005. The iterations were terminated at this point due to the smallness of the adjustment.

During the calibration using the Gauss-Newton method, ρ was assumed as 0.75 during the first two iterations, and 1.0 afterwards. Larger values of ρ were avoided during first few iterations to avoid crashing the model. The Gauss-Newton method required the least amount of computer time and the fastest rate of convergence.

The minimax method is the most time consuming because of the linear programming (LP) algorithm. The computer time requirement increased with the number of time steps in the simulation runs because it increased the size of the linear programming problem. A Sparc 1+ computer using the IMSL package needed 10-60 minutes of computer time for a single iteration of the minimax, when compared with split seconds needed by other methods. Table 3 shows that parameter values are approximately the same for the most important parameter group m_1 consisting of $n_1 \ldots n_6$. Plots of the observed and simulated water levels using all three methods do not show any visually detectable difference.

CONCLUSIONS

SVD is useful in identifying and understanding an arbitrary number of parameters when the equations involved are overdetermined, even-determined, underdetermined or singular. Calibration of the 27 parameters in the Niagara river model using different numbers of parameter groups show that the model has a maximum of 7 independent parameter groups, and the optimum parameter dimension is about 3.

The study shows that the underdetermined identification problem with 27 parameters can be reduced to an even-determined or an over-determined problem by creating a number of parameter groups equal to the rank of the sensitivity matrix (7) or less. The groups can be created using the information from parameter space resolution matrices, correlation matrices, singular vectors or the geometrical layout of the river system. Parameters corresponding to river reaches between gages appear as groups in the correlation and resolution matrices, and therefore can be considered as groups. Once parameter groups are created such that the problem is neither underdetermined nor singular, optimization methods such as the Gauss Newton and minimax methods can be used to carry out the same calibration.

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APPENDIX II. NOTATION

The following symbols are used in the paper

$\mathbf{A}^k$	sensitivity matrix $[a^k_{j,i}]$
A	cross sectional area of a channel
$\mathbf{A}_s$	$\sum_{k=1}^{l} \mathbf{A}^k$
a^k_{ji}	sensitivity of j th water level with respect to i th parameter at time k.
$\mathbf{d}$	intermediate variable defined as $\mathbf{U}^T \varepsilon_s$ which give linear combinations of differences in water levels
h_i	water level at gage i
k	time step number
l	number of time steps
m	number of observation stations
m_j	group values of handpicked parameters
n	number of parameters
n_b	Manning's coefficient
n_j	Manning's coefficient of river section j
n^*_j	new parameter combination obtained using the SVD method
Q	discharge in a channel
q	rank of the matrix, or number of singular values used
r	number of iterations
R	hydraulic radius
$\mathbf{R}$	parameter resolution matrix
$\mathbf{U}$, $\mathbf{V}$	$m \times m$ and $n \times n$ matrices of orthogonal singular vectors
x_i	a generic symbol for parameter i
Y^k_j	observed water level at node j at time k (m)
y^k_j	simulated water level at node j at time k (m)
$\Delta\mathbf{x}$	vector of parameter corrections $(n_1, n_2 \ldots)^T$
ε_s	vector consisting of summations of errors
ε_{sj}	summation of errors at gage j over time

Λ diagonal matrix of singular values

λ_i i th singular value

$\rho_{i,j}$ correlation among parameter corrections

τ_b shear stress at channel bottom

Υ^2 covariance of parameter corrections

Subscripts

s summation over time

r iteration number

Superscripts

k value at time step k

Figures

Fig 1: Map of the upper Niagara River.

Fig 2: Sensitivity matrix (Max diameter=31519); gages G3-G9 are marked as 1-7.

Fig 3: Convergence of parameters and reduction of errors with iterations.

Fig 4: Observed and simulated water levels after calibration.

Fig 5: Parameter resolution matrix (Max. Diameter = 0.96).

Fig 6: Correlation matrix (Max diameter = 1.0).

Fig 7: Variation of output error and parameter uncertainty with parameter dimension.

Table 1: Manning's Roughness Coefficients obtained using SVD

Param.	1	2	3	4	5	6	7	8	9
Indiv.	.0273	.0321	.0352	.0280	.0226	.0239	.0237	.0208	.0248
7 groups	.0329	.0329	.0329	.0329	.0329	.0226	.0226	.0226	.0229
3 groups	.0327	.0327	.0327	.0327	.0327	.0220	.0220	.0220	.0238
Param.	10	11	12	13	14	15	16	17	18
Indiv.	.0267	.0258	.0211	.0286	.0269	.0099	.0166	.0203	.0200
7 groups	.0229	.0229	.0229	.0263	.0263	.0263	.0194	.0194	.0194
3 groups	.0238	.0238	.0238	.0238	.0238	.0238	.0220	.0220	.0220
Param.	19	20	21	22	23	24	25	26	27
Indiv.	.0239	.0268	.0284	.0277	.0133	.0183	.0346	.0255	.0223
7 groups	.0261	.0261	.0261	.0261	.0261	.0261	.0261	.0261	.0007
3 groups	.0238	.0238	.0238	.0238	.0238	.0238	.0238	.0238	.0238

KEY WORDS

Bed roughness calibration, canal network, Open chanel flow Hydraulics, Inverse problem, parameter identification, Singular value decomposition, least square, Gauss-Newton, Linear Programming,

Table 2: Parameter groups formed by combining individual reach parameters; all groups are combinations of the 7 basic groups in column 2

Groups of Original Parameters	Number of par. groups and formation					
	7	5	4	3	2	1
$n_1 \ldots n_5$	m_1	m_1	m_1	m_1	m_1	m_1
$n_6 \ldots n_8$	m_2	m_2	m_2	m_2	m_2	m_1
$n_9 \ldots n_{12}$	m_3	m_3	m_3	m_3	m_2	m_1
$n_{13} \ldots n_{15}$	m_4	m_4	m_3	m_3	m_2	m_1
$n_{16} \ldots n_{18}$	m_5	m_5	m_4	m_2	m_2	m_1
$n_{19} \ldots n_{26}$	m_6	m_5	m_1	m_3	m_2	m_1
n_{27}	m_7	m_5	m_3	m_3	m_2	m_1

Table 3: Results of the calibration of 7 handpicked parameter groups using different methods

Group	Min. Bias	Least Sqr.	Minimax
1	0.03289	0.03200	0.03296
2	0.02270	0.02539	0.02558
3	0.02282	0.02546	0.02504
4	0.02615	0.02504	0.02239
5	0.01948	0.02570	0.01816
6	0.02948	0.02541	0.02371
7	0.00701	0.02501	0.02408
Err. Variance (m^2)	0.00187	0.00959	0.05361
Mean Sq. Bias (m^2	0.00072	0.00848	0.05332

Parameter	Individual	7 groups	3 groups
1	0.0273	0.0329	0.0327
2	0.0321	0.0329	0.0327
3	0.0352	0.0329	0.0327
4	0.0280	0.0329	0.0327
5	0.0226	0.0329	0.0327
6	0.0239	0.0226	0.0220
7	0.0237	0.0226	0.0220
8	0.0208	0.0226	0.0220
9	0.0248	0.0229	0.0238
10	0.0267	0.0229	0.0238
11	0.0258	0.0229	0.0238
12	0.0211	0.0229	0.0238
13	0.0286	0.0263	0.0238
14	0.0269	0.0263	0.0238
15	0.0099	0.0263	0.0238
16	0.0166	0.0194	0.0220
17	0.0203	0.0194	0.0220
18	0.0200	0.0194	0.0220
19	0.0239	0.0261	0.0238
20	0.0268	0.0261	0.0238
21	0.0284	0.0261	0.0238
22	0.0277	0.0261	0.0238
23	0.0133	0.0261	0.0238
24	0.0183	0.0261	0.0238
25	0.0346	0.0261	0.0238
26	0.0255	0.0261	0.0238
27	0.0223	0.0007	0.0238